J. Fluid Mech. (1985), *vol.* 150, *pp.* 417–426
Printed in Great Britain

Waves at the interface between a clear liquid and a mixture in a two-phase flow

By PHOOLAN PRASAD

Department of Applied Mathematics, Indian Institute of Science, Bangalore-560012, India

(Received 14 June 1983 and in revised form 2 July 1984)

In the two-dimensional sedimentation process beneath an inclined wall, the mixture of the particulate and liquid phases is separated from the wall by a boundary layer of the clear liquid. This paper contains a simple mathematical model giving waves on the interface between the clear liquid and the mixture. These waves are caused by a discontinuity in the gradient of the tangential velocity of the clear liquid, across the interface. In the limiting case of small concentration of the particulate phase in the mixture, the model gives a dispersive wave running upward along the interface in the direction of the flow in the boundary layer. The effect of finite concentration is to introduce a damping.

1. Introduction

Consider the two-dimensional problem of sedimentation beneath a plane inclined wall, as shown in figure 1. When the particle concentration α_0 in the mixture is smaller than a critical value α_a, it is possible to get a solution (Schneider 1982) in which the mixture, with constant concentration α_0, is separated from the clear liquid by a horizontal kinematic shock S_1 and from the sediment at the bottom by another horizontal kinematic shock S_2 (existence of the critical value α_a was predicted by Kynch (1952), who first formulated a comprehensive mathematical theory of sedimentation). Moreover, the mixture is also separated from the wall by a boundary layer of clear liquid. If α_0 is small the surface S_2 rises slowly, and the flow of the clear liquid in the boundary layer can be assumed to be steady. When experiments are done to simulate such a flow, it has been observed (Schaflinger 1983) that waves of deformation travel rapidly upward along the surface of separation. In this paper we shall show that a simple model of the two-phase flow in the mixture does give a surface wave propagating upward along the interface. In the limiting case when the concentration α_0 of the particulate phase tends to zero, this wave is purely dispersive. The effect of small α_0 is to introduce a small damping in this wave. However, these waves propagate too rapidly to be observed in the experiments reported by Schaflinger (1983). The dispersion relation shows that our waves are stable. The experimentally observed waves of Schaflinger are instabilities at a different timescale (see Herbolzheimer 1983; Davis, Herbolzheimer & Acrivos 1983).

In the basic flow of the sedimentation process below the inclined wall there is no discontinuity in the tangential velocity of the liquid across the interface. The surface waves obtained in this paper are caused by a discontinuity in the gradient (in the direction normal to the interface) of the tangential velocity. Also, gravity does not have any effect on these waves in the first approximation. Let us describe a model of the single-phase flow in which these waves can also be obtained. Consider a two-dimensional motion of an incompressible liquid of density ρ_f such that the liquid

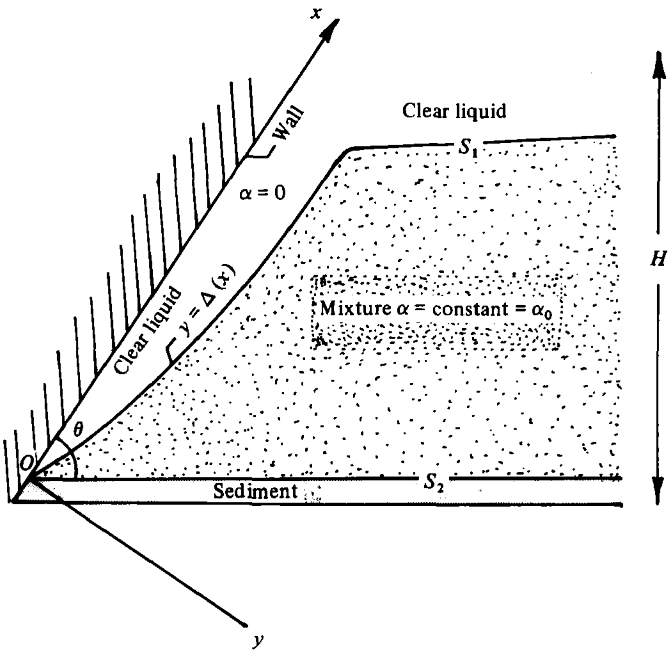

FIGURE 1. The basic undisturbed solution when $\alpha < \alpha_a$.

is at rest in the domain $y < 0$, has a shearing motion in the x-direction in the domain $y > 0$ given by the velocity distribution $(u, v) = (y\bar{u}_{0y}, 0)$, where $\bar{u}_{0y}$ is constant, and is bounded at the top by the plane $y = \Delta$. Then the plane $y = 0$ is an interface separating the liquid at rest from the liquid in uniform shearing motion. We create small perturbations on this flow and use the usual boundary conditions that the perturbed interface is a stream surface, the normal velocity on the plane $y = \Delta$ is zero, and the velocity components vanish as $y \rightarrow -\infty$. From linear analysis of waves of the form $e^{i(\bar{k}x-\bar{\omega}t)}$, we can deduce the following dispersion relation:

$$\frac{\bar{\omega}}{\bar{k}} = \bar{u}_{0y} \frac{\tanh \bar{k}\Delta}{\bar{k}(1+\tanh \bar{k}\Delta)} = \bar{V}_{ph}, \text{ say,} \tag{1.1}$$

which represents a dispersive wave. Now we proceed to show how these waves appear in the sedimention process and how they are modified owing to finite concentration α of the particulate phase in the mixture.

2. Basic equations

We choose the origin O on the surface S_2, which is assumed to be fixed for small α, the x-axis upward along the wall, and the y-axis perpendicular to it sloping downward (see figure 1). Let θ be the angle between the inclined wall of the vessel and the horizontal direction. We neglect viscosity in the equations of motion for the clear liquid as well as the mixture. Let $y = \delta(x, t)$ be the equation of the perturbed interface. We denote the components of the velocity and the pressure in the clear liquid domain $0 < y < \delta(x, t)$ by (u, v) and p respectively. In the mixture the corresponding quantities for the solid phase, fluid phase and the mixture as a whole will be denoted by subscripts s, f and m respectively. We take the force of interaction between the particulate and the liquid phases to be proportional to the relative velocity, i.e. equal to the vector $F(\alpha)(u_f-u_s,\ v_f-v_s)$, where F is a function of the concentration α and depends also on some other physical parameters of the flow. The expression for $F(\alpha)$ can be derived using equations (1.39), (4.9) and (8.2) of Wallis (1969) for the steady sedimentation process in a vertical tube by setting F_{12} (of Wallis) equal to $-F(\alpha)\ (u_f-u_s)$. We finally get

$$F(\alpha) = \frac{g\alpha(\rho_s-\rho_f)}{U(1-\alpha)^{n-2}}, \quad n = \text{constant}. \tag{2.1}$$

Here ρ_s and ρ_f are the densities of the particles and the liquid, U is the terminal settling velocity of a single particle in the liquid at rest and g the acceleration due to gravity. The constant n depends on a suitably defined Reynolds number (Wallis 1969, p. 178).

For non-dimensionalization we use the wavelength L and wave velocity S of the surface wave. The non-dimensional variables, denoted by ξ, η, τ, $\tilde{u}$, $\tilde{v}$, $\tilde{p}$, $\tilde{\delta}$, $\tilde{F}$, are defined by

$$\left.\begin{aligned} &\xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \tau = \frac{St}{L}, \quad \tilde{u} = \frac{u}{S}, \quad \tilde{v} = \frac{v}{S}, \\ &\tilde{p} = \frac{p}{\rho_f S^2}, \quad \tilde{\delta} = \frac{\delta}{L}, \quad \tilde{F} = \frac{L}{S\rho_f}F, \quad \gamma = \frac{\rho_s}{\rho_f}. \end{aligned}\right\} \tag{2.2}$$

In terms of the non-dimensional variables, the equations of motion are

for the clear liquid in $\tilde{\eta} < \tilde{\delta}$

$$\tilde{u}_\xi + \tilde{v}_\eta = 0, \tag{2.3}$$

$$\tilde{u}_\tau + \tilde{u}\tilde{u}_\xi + \tilde{v}\tilde{u}_\eta = -\tilde{p}_\xi - \frac{gL}{S^2}\sin\theta, \tag{2.4}$$

$$\tilde{v}_\tau + \tilde{u}\tilde{v}_\xi + \tilde{v}\tilde{v}_\eta = -\tilde{p}_\eta + \frac{gL}{S^2}\cos\theta; \tag{2.5}$$

for the particulate phase in the mixture $\eta > \tilde{\delta}$

$$\alpha_\tau + (\alpha\tilde{u}_s)_\xi + (\alpha\tilde{v}_s)_\eta = 0, \tag{2.6}$$

$$\tilde{u}_{s\tau} + \tilde{u}_s\tilde{u}_{s\xi} + \tilde{v}_s\tilde{u}_{s\eta} = -\frac{1}{\gamma}\tilde{p}_{m\xi} + \frac{\tilde{F}}{\alpha}\frac{1}{\gamma}(\tilde{u}_f - \tilde{u}_s) - \frac{gL}{S^2}\sin\theta, \tag{2.7}$$

$$\tilde{v}_{s\tau} + \tilde{u}_s\tilde{v}_{s\xi} + \tilde{v}_s\tilde{u}_{s\eta} = -\frac{1}{\gamma}\tilde{p}_{m\eta} + \frac{\tilde{F}}{\alpha}\frac{1}{\gamma}(\tilde{v}_f - \tilde{v}_s) + \frac{gL}{S^2}\cos\theta; \tag{2.8}$$

for the liquid phase in the mixture $\eta > \tilde{\delta}$

$$-\alpha_\tau + \{(1-\alpha)\tilde{u}_f\}_\xi + \{(1-\alpha)\tilde{v}_f\}_\eta = 0, \tag{2.9}$$

$$\tilde{u}_{f\tau} + \tilde{u}_f\tilde{u}_{f\xi} + \tilde{v}_f\tilde{u}_{f\eta} = -\tilde{p}_{m\xi} - \frac{\tilde{F}}{1-\alpha}(\tilde{u}_f - \tilde{u}_s) - \frac{gL}{S^2}\sin\theta, \tag{2.10}$$

$$\tilde{v}_{f\tau} + \tilde{u}_f\tilde{v}_{f\xi} + \tilde{v}_f\tilde{v}_{f\eta} = -\tilde{p}_{m\eta} - \frac{\tilde{F}}{1-\alpha}(\tilde{v}_f - \tilde{v}_s) + \frac{gL}{S^2}\cos\theta. \tag{2.11}$$

The boundary conditions on the interface $\eta = \tilde{\delta}(\xi,\tau)$ are

the interface is a stream surface for the particulate phase in the mixture, i.e.

$$\tilde{\delta}_\tau + \tilde{u}_s\tilde{\delta}_\xi - \tilde{v}_s = 0 \quad \text{on } \eta = \tilde{\delta}(\xi,\tau); \tag{2.12}$$

the volume flux of liquid across the interface is continuous, i.e.

$$(1-\alpha)(\tilde{\delta}_\tau+\tilde{u}_f\tilde{\delta}_\xi-\tilde{v}_f) = \tilde{\delta}_\tau+\tilde{u}\tilde{\delta}_\xi-\tilde{v} \quad \text{on } \eta = \tilde{\delta}(\xi,\tau); \tag{2.13}$$

the pressure jump $p_m - p$ across the interface is equal to the jump in the relative flux of the liquid, i.e.

$$(1+\tilde{\delta}_\xi^2)(\tilde{p}_m-\tilde{p}) = -(1-\alpha)(\tilde{\delta}_\tau+\tilde{u}_f\tilde{\delta}_\xi-\tilde{v}_f)^2+(\tilde{\delta}_\tau+\tilde{u}\tilde{\delta}_\xi-\tilde{v})^2 \quad \text{on } \eta = \tilde{\delta}(\xi,\tau). \tag{2.14}$$

Finally, the normal component of the velocity in the clear liquid vanishes at the wall $\eta = 0$, i.e.

$$\tilde{v}(\xi,0,\tau) = 0, \tag{2.15}$$

and the perturbation of all quantities in the mixture tends to zero as $\eta \to \infty$.

Let $y = \Delta(x)$ or $\eta = \tilde{\delta}_0(\xi)$ be the equation to the interface in the basic steady solution of Schneider (1982). The problem has three lengthscales: (i) H, the height of the vessel; (ii) L, the wavelength of the waves; (iii) Δ_0, a measure of the thickness of the boundary layer of clear liquid. The problem has three velocity scales also: (i) U, the terminal settling velocity of a single particle; (ii) S, the velocity of the waves; (iii) U_B, a measure of the velocity of the clear liquid in the x-direction in the boundary layer. In Schneider's solution the ratios Δ_0/H and U/U_B are both small and of the same order as that of $Gr^{-\frac{1}{3}}Re$, where Gr is a sedimentation Grashof number and Re a sedimentation Reynolds number, defined by

$$Gr = \frac{H^3 g\alpha_0(\rho_s-\rho_f)}{\rho_f \nu_f^2}, \quad Re = \frac{HU}{\nu_f}, \tag{2.16a}$$

where ν_f is the kinematic viscosity of the liquid. In the case of the surface-wave phenomena we get another non-dimensional parameter $\epsilon = U/S$, which we assume to be small. Our mathematical model is based on the assumption that ϵ and $Gr^{-\frac{1}{3}}Re$ are both of the same order of smallness, i.e.

$$\epsilon = \frac{U}{S} = O(Gr^{-\frac{1}{3}}Re). \tag{2.16b}$$

Since we shall use Schneider's solution in the clear-liquid boundary layer, we mention here that his solution is derived on two basic assumptions, namely

$$\frac{Gr}{Re^2} \to \infty \quad \text{and} \quad \frac{Gr}{Re^4} \to 0.$$

The y-component of the velocity in the boundary layer in Schneider's solution is of the same order as U. Assuming that there are quite a large number of waves on the interface, we find that the basic undisturbed solution of the sedimentation process (denoted by a subscript 0 on the variables) can be taken to be a slowly varying function of ξ. To account for this we introduce a slow variable

$$\xi_1 = \epsilon\xi. \tag{2.17}$$

The situation is similar to the study of stability of a non-parallel boundary layer (Saric & Nayfeh 1975). The quantities in the unperturbed state are given as

$$\begin{bmatrix} \tilde{u}_0 \\ \tilde{v}_0 \\ \tilde{\delta}_0 \\ \tilde{\alpha}_0 \\ \tilde{p}_0 \\ \tilde{p}_{m0} \\ \tilde{u}_{s0} \\ \tilde{v}_{s0} \\ \tilde{u}_{f0} \\ \tilde{v}_{f0} \end{bmatrix} = \begin{bmatrix} u_0(\xi_1,\eta) \\ 0 \\ \delta_0(\xi_1) \\ \alpha_0 \\ p_{m0}(\xi_1) \\ p_{m0}(\xi_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} u_0^{(1)}(\xi_1,\eta) \\ v_0(\xi_1,\eta) \\ \delta_0^{(1)}(\xi_1,\eta) \\ \alpha_0^{(1)}(\xi_1,\eta) \\ p_{m0}^{(1)}(\xi_1,\eta) \\ p_{m0}^{(1)}(\xi_1,\eta) \\ u_{s0} \\ v_{s0} \\ u_{f0} \\ v_{f0} \end{bmatrix} + \epsilon^2 \begin{bmatrix} u_0^{(2)}(\xi_1,\eta) \\ v_0^{(1)}(\xi_1,\eta) \\ \delta_0^{(2)}(\xi_1,\eta) \\ \alpha_0^{(2)}(\xi_1,\eta) \\ p_0^{(2)}(\xi_1,\eta) \\ p_{m0}^{(2)}(\xi_1,\eta) \\ u_{s0}^{(1)}(\xi_1,\eta) \\ v_{s0}^{(1)}(\xi_1,\eta) \\ u_{f0}^{(1)}(\xi_1,\eta) \\ v_{f0}^{(1)}(\xi_1,\eta) \end{bmatrix} + \ldots \quad (2.18)$$

Here $\alpha_0, u_{s0}, v_{s0}, u_{f0}, v_{f0}$ are constants, $p_{m0}(\xi_1)+\epsilon p_{m0}^{(1)}(\xi_1,\eta)$ represents the hydrostatic pressure in the mixture and is also equal to the first two terms of $\tilde{p}_0$, and these together with $\delta_0(\xi_1)$, $u_0(\xi_1,\eta)$ and $v_0(\xi_1,\eta)$ constitute Schneider's approximate solution for the basic flow.

For the perturbation of the basic flow we set

$$\left.\begin{aligned} \tilde{u} = \tilde{u}_0+\epsilon u', \quad \tilde{v} = \epsilon(\tilde{v}_0+v'), \quad \tilde{p} = \tilde{p}_0+\epsilon p', \quad \tilde{\delta} = \tilde{\delta}_0+\epsilon\delta', \\ \tilde{p}_m = \tilde{p}_{m0}+\epsilon p'_m, \quad \alpha = \tilde{\alpha}_0+\epsilon\alpha', \quad \tilde{u}_s = \tilde{u}_{s0}+\epsilon u'_s, \\ \tilde{v}_s = \tilde{v}_{s0}+\epsilon v', \quad \tilde{u}_f = \tilde{u}_{f0}+\epsilon u'_f, \quad \tilde{v}_f = \tilde{v}_{f0}+\epsilon v'_f. \end{aligned}\right\} \quad (2.19)$$

From (2.18) and (2.19) we find that the surface wave induces a perturbation in the velocity components in the mixture comparable to the unperturbed velocity components. Substituting (2.19) in (2.3)–(2.15), taking into account the equations satisfied by the basic flow and retaining only the dominant terms in ϵ, we get

in $\eta \leqslant \delta_0$,

$$u'_\xi+v'_\eta = 0 \quad (2.20)$$

$$u'_\tau+u_0 u'_\xi+u_{0\eta} v' = -p'_\xi, \quad (2.21)$$

$$v'_\tau+u_0 v'_\xi = -p'_\eta; \quad (2.22)$$

in $\eta \geqslant \delta_0$,

$$\alpha'_\tau+\alpha_0 u'_{s\xi}+\alpha_0 v'_{s\eta} = 0, \quad (2.23)$$

$$u'_{s\tau} = -\frac{1}{\gamma}p'_{m\xi}+\frac{\bar{F}(\alpha_0)}{\gamma\alpha_0}(u'_f-u'_s), \quad (2.24)$$

$$v'_{s\tau} = -\frac{1}{\gamma}p'_{m\eta}+\frac{\bar{F}(\alpha_0)}{\gamma\alpha_0}(v'_f-v'_s), \quad (2.25)$$

$$-\alpha'_\tau+(1-\alpha_0)u'_{f\xi}+(1-\alpha_0)v'_{f\eta} = 0, \quad (2.26)$$

$$u'_{f\tau} = -p'_{m\xi}-\frac{\bar{F}(\alpha_0)}{1-\alpha_0}(u'_f-u'_s), \quad (2.27)$$

$$v'_{f\tau} = -p'_{m\eta}-\frac{\bar{F}(\alpha_0)}{1-\alpha_0}(v'_f-v'_s); \quad (2.28)$$

on the interface $\eta = \delta_0(\xi_1)$,

$$\delta'_\tau - v'_s = 0, \tag{2.29}$$

$$\alpha_0 \delta'_\tau + u_0 \delta'_\xi + (1-\alpha_0) v'_f - v' = 0, \tag{2.30}$$

$$p'_m = p'; \tag{2.31}$$

and finally, on the wall $\eta = 0$,

$$v'(\xi, 0, \tau) = 0. \tag{2.32}$$

3. Derivation of the dispersion relation

The quantities in the basic flow (denoted by subscripts 0), which appear in the coefficients of (2.20)–(2.32), are functions of the slow variable ξ_1 and are actually to be treated as constants. Thus in the analysis of the equations for perturbations, in the first approximation, we proceed as if we have a parallel flow in the boundary layer. The perturbation variables u', v', ... should also be expanded in positive powers of ϵ. However, our interest here is only in the dispersion relation; hence we neglect all terms containing higher powers of ϵ and write

$$u' = U_1(\xi_1, \eta)\,\mathrm{e}^{\mathrm{i}\phi}, \quad \alpha' = A_1(\xi_1, \eta)\,\mathrm{e}^{\mathrm{i}\phi}, \quad \delta' = D_1(\xi_1)\,\mathrm{e}^{\mathrm{i}\phi} \quad \text{etc.}, \tag{3.1}$$

where the phase function ϕ is of the form

$$\phi = k(\xi_1)\,\xi - \omega\tau. \tag{3.2}$$

Substituting (3.1) in (2.20)–(2.22) and eliminating U_1 and P_1, we get the inviscid Orr–Sommerfeld equation

$$\left\{u_0(\eta) - \frac{\omega}{k}\right\}\left(\frac{\mathrm{d}^2 V_1}{\mathrm{d}\eta^2} - k^2 V_1\right) - u_{0\eta\eta} V_1 = 0. \tag{3.3}$$

In the boundary layer of the sedimentation process, the function $u_0(\eta)$ is a linear function of η (Schneider 1982, A.4 Example) so that (3.3) becomes $\mathrm{d}^2 V_1/\mathrm{d}\eta^2 - k^2 V_1 = 0$. The boundary condition for V_1 is obtained from (2.32) as $V_1(\eta = 0) = 0$. Solving for V_1 and using the relation between V_1 and P_1, we get

$$P_1(\eta) = \mathrm{i}\frac{V_1(\delta_0)}{k \sinh k\delta_0}\{(\omega - u_0 k)\cosh k\eta + u_{0\eta} \sinh k\eta\}, \tag{3.4}$$

which gives the variation with respect to η of the pressure in the clear-liquid boundary layer.

Now we turn to (2.23)–(2.28) in the mixture. Differentiating (2.24) with respect to ξ, (2.25) with respect to η, adding and then eliminating the combinations $u_{s\xi} + v_{s\eta}$ and $u_{f\xi} + v_{f\eta}$ with the help of (2.23) and (2.26) respectively, we get an expression for $\Delta p'_m$ in terms of temporal derivatives of α'. Similarly, starting from (2.27) and (2.28) we get another expression for $\Delta p'_m$. Equating these two expressions for $\Delta p'_m$, we get an equation for α':

$$\left\{\frac{\tilde{F}(\alpha_0)}{\alpha_0(1-\alpha_0)} + [\gamma(1-\alpha_0) + \alpha_0]\frac{\partial}{\partial\tau}\right\}\frac{\partial\alpha'}{\partial\tau} = 0. \tag{3.5}$$

The first term of the operator in (3.5) corresponds to an exponential decay of α' with decay time $\alpha_0(1-\alpha_0)\{\gamma(1-\alpha_0)+\alpha_0\}/\tilde{F}(\alpha_0)$, which is comparable to the time period of the wave. Therefore the solution of (3.5) relevant to the wave motion is given by $\partial\alpha'/\partial\tau = 0$. Using $\partial\alpha'/\partial\tau = 0$ in the relations between α' and p'_m mentioned above, we find that p'_m satisfies the Laplace equation. Substituting the expression of the form

(3.1) for p'_{m}, i.e. $p'_{\mathrm{m}} = P_{\mathrm{m}}(\eta)\,\mathrm{e}^{\mathrm{i}\phi}$, we find that the solution vanishing at infinity, $\eta \to \infty$, is

$$P_{\mathrm{m1}} = P_{\mathrm{m1}}(\delta_0)\,\mathrm{e}^{-k(\eta-\delta_0)}. \tag{3.6}$$

Variations with respect to η for all other variables, vanishing at infinity, $\eta \to \infty$, in the mixture $\eta \geqslant \delta_0$, are given by

$$[U_{\mathrm{s1}}(\eta), V_{\mathrm{s1}}(\eta), U_{\mathrm{f1}}(\eta), V_{\mathrm{f1}}(\eta)] = [U_{\mathrm{s1}}(\delta_0), V_{\mathrm{s1}}(\delta_0), U_{\mathrm{f1}}(\delta_0), V_{\mathrm{f1}}(\delta_0)]\,\mathrm{e}^{-k(\eta-\delta_0)}, \tag{3.7}$$

where
$$U_{\mathrm{f1}}(\delta_0) = KU_{\mathrm{s1}}(\delta_0), \quad V_{\mathrm{s1}}(\delta_0) = \mathrm{i}U_{\mathrm{s1}}(\delta_0), \quad V_{\mathrm{f1}}(\delta_0) = \mathrm{i}KU_{\mathrm{s1}}(\delta_0) \tag{3.8}$$

and
$$P_{\mathrm{m1}}(\delta_0) = \frac{\omega}{k}\,\frac{\gamma\Gamma/(1-\alpha_0) - \mathrm{i}\gamma\omega - (\gamma-1)\Gamma}{\Gamma/(1-\alpha_0) - \mathrm{i}\omega}\,U_{\mathrm{s1}}(\delta_0), \tag{3.9}$$

with
$$\Gamma = \frac{\tilde{F}(\alpha_0)}{\alpha_0} = \frac{Lg}{SU}\,\frac{\rho_{\mathrm{s}}-\rho_{\mathrm{f}}}{\rho_{\mathrm{f}}(1-\alpha)^{n-1}}. \tag{3.10}$$

Substituting expressions of the form (3.1) in the boundary conditions (2.29)–(2.31), we get

$$V_{\mathrm{s1}}(\delta_0) + \mathrm{i}\omega D_1 = 0, \tag{3.11}$$

$$\mathrm{i}(\alpha_0\,\omega - u_0\,k)\,D_1 = (1-\alpha_0)\,V_{\mathrm{f1}}(\delta_0) - V_1(\delta_0) \tag{3.12}$$

and
$$P_{\mathrm{m1}}(\delta_0) = P_1(\delta_0). \tag{3.13}$$

Eliminating V_{s1} and U_{s1} between (3.9), (3.11) and $V_{\mathrm{s1}}(\delta_0) = \mathrm{i}U_{\mathrm{s1}}(\delta_0)$, we get an expression for $P_{\mathrm{m1}}(\delta_0)$ in terms of D_1. Again, eliminating $V_1(\delta_0)$, V_{f1} and V_{s1} between (3.4), (3.11), (3.12) and $V_{\mathrm{s1}}(\delta_0) = KV_{\mathrm{f1}}(\delta_0)$, we get an expression for $P_1(\delta_0)$:

$$P_1(\delta_0) = \frac{1}{k}\{[(1-\alpha_0)\,K + \alpha_0]\,\omega - u_0\,k\}\,\{(\omega - u_0\,k)\coth k\delta_0 + u_{0\eta}\}\,D_1. \tag{3.14}$$

Substituting the expression for $P_{\mathrm{m}}(\delta_0)$ in terms of D_1 and the expression (3.14) in (3.13), we get the following dispersion relation:

$$\begin{aligned}
&\{\gamma + (\coth k\delta_0)\,(\gamma - \alpha_0(\gamma-1))\}\,\omega^3 + \Big\{-[u_0\,k\coth k\delta_0 \\
&\quad + (u_0\,k\coth k\delta_0 - u_{0\eta})(\gamma - \alpha_0(\gamma-1))] + \frac{\Gamma}{1-\alpha_0}(1-\alpha_0+\gamma\alpha_0+\coth k\delta_0)\,\mathrm{i}\Big\}\,\omega^2 \\
&\quad + \Big\{(u_0\,k\coth k\delta_0 - u_{0\eta})\,u_0\,k - \frac{\Gamma}{1-\alpha_0}(2u_0\,k\coth k\delta_0 - u_{0\eta})\,\mathrm{i}\Big\}\,\omega \\
&\quad + \frac{\Gamma}{1-\alpha_0}\,u_0\,k(u_0\,k\coth k\delta_0 - u_{0\eta})\,\mathrm{i} = 0 \quad \text{on } \eta = \delta_0.
\end{aligned} \tag{3.15}$$

The dispersion relation written here looks quite complicated owing to the presence of $u_0(\delta_0)$, which, if non-zero, implies a discontinuity in the tangential velocity of the fluid across the interface. However, in Schneider's solution the tangential velocity vanishes at the interface, i.e. $u_0(\delta_0) = 0$, so that either $\omega = 0$ for all values of k or

$$\begin{aligned}
&\{\gamma + \coth k\delta_0(\gamma + \alpha_0 - \alpha_0\,\gamma)\}\,\omega^2 \\
&\quad + \Big\{(\gamma + \alpha_0 - \gamma\alpha_0)\,u_{0\eta} + \frac{\Gamma}{1-\alpha_0}(1-\alpha_0+\gamma\alpha_0+\coth k\delta_0)\,\mathrm{i}\Big\}\,\omega + \frac{\Gamma}{1-\alpha_0}\,u_{0\eta}\,\mathrm{i} = 0.
\end{aligned} \tag{3.16}$$

The case $\omega = 0$ for all k is of no relevance to us, since the waves under consideration have non-zero velocity. The other two modes of the dispersion relation given by (3.16) are still too complicated to give physically simple results.

In most of the experiments where these waves might be observed, α_0 would be small.

Hence we first take the limiting case $\alpha_0 \to 0$, when (3.16) becomes

$$\gamma(1+\coth k\delta_0)\,\omega^2+\{\gamma u_{0\eta}+\Gamma_0(1+\coth k\delta_0)\,\mathrm{i}\}\,\omega+\Gamma_0\,u_{0\eta}\,\mathrm{i}=0 \tag{3.17}$$

where, from (3.10),

$$\Gamma_0=\lim_{\alpha\to 0}\Gamma=\frac{Lg}{SU}\frac{\rho_s-\rho_f}{\rho_f}. \tag{3.18}$$

Equation (3.17) gives two simple and elegant roots for ω:

$$\omega_1=\frac{-u_{0\eta}}{1+\coth\delta_0 k},\quad \omega_2=-\frac{\Gamma_0}{\gamma}\mathrm{i}. \tag{3.19}$$

The first mode given by the real root ω_1 represents a dispersive wave without any damping, and since $-u_{0\eta}$ is positive these waves move upward. It is the same wave as given by (1.1) (the $\bar{y}$-axis for (1.1) is in the negative η-direction) for the simple model of a one-phase flow. The second mode, given by the purely imaginary root ω_2 in (3.19), represents a non-propagating disturbance, and since $-\mathrm{i}\omega_2=-\Gamma_0/\gamma<0$, this mode decays in time.

4. A few properties of the waves at the interphase

For the dispersive wave the phase velocity V_{ph} and group velocity V_{gr} are

$$V_{\mathrm{ph}}\equiv\frac{\omega_1}{k}=\frac{-u_{0\eta}\tanh k\delta_0}{k(1+\tanh k\delta_0)},\quad V_{\mathrm{gr}}=-u_{0\eta}\,\delta\frac{1-\tanh k\delta_0}{1+\tanh k\delta_0}, \tag{4.1}$$

and they satisfy $V_{\mathrm{ph}}-V_{\mathrm{gr}}>0$ for $k>0$. In the long-wave limit $k\delta_0\to 0$ we get

$$\lim_{k\delta_0\to 0}V_{\mathrm{ph}}=-u_{0\eta}\,\delta_0=\lim_{k\delta_0\to 0}V_{\mathrm{gr}}, \tag{4.2}$$

i.e. both the phase velocity and group velocity tend to the same finite non-zero limit. In the short-wave limit $k\delta_0\to\infty$ both tend to zero.

For small $k\delta_0$,

$$\omega=-u_{0\eta}\{k\delta_0-k^2\delta_0^2-\tfrac{1}{3}k^3\delta_0^3+\ldots\}. \tag{4.3}$$

The presence of the quadratic term in $k\delta_0$ on the right-hand side of (4.3) (in the case of surface water waves or gravity waves this quadratic term is absent) shows that the dispersion of the long wave here is much stronger than that for the surface water waves under gravity.

In the basic equations for the perturbation or in the dispersion relation (3.15), or (3.16) or (3.17), the acceleration g due to gravity does not appear directly; it appears only in the parameter Γ through the coefficient $F(\alpha)$ (see (2.1)) of the force of interaction. Therefore, even though the basic sedimentation process is caused by gravity, these waves are not. These waves are produced by the discontinuity in the normal derivative of the tangential component of the velocity at the interface.

The two complex roots of (3.16) can easily be written down. But these are not very useful, and hence to find the effect of a finite value of the particle concentration α_0 we need to evaluate the roots numerically for various values of the parameters γ (the density ratio of the two phases), Γ and $u_{0\eta}$. For all permissible values of the parameters γ and $u_{0\eta}$ the effect of non-zero α_0 is to introduce a small damping to the dispersive wave and a very small change in the phase velocity. Tables 1 and 2 give the values of the phase velocity $\omega_{1\mathrm{R}}/k$ and damping $\omega_{1\mathrm{I}}$ ($\omega_1=\omega_{1\mathrm{R}}+\mathrm{i}\omega_{1\mathrm{I}}$) for $-u_{0\eta}=1$, $\gamma=2$, $\delta_0=1$, $\Gamma=1$ for various values of α. Significant variation in the phase velocity and damping of the first mode takes place only for very large or small values of the parameters $-u_{0\eta}$, γ and Γ.

$k\backslash\alpha_0$	0.001	0.05	0.1	0.5
0.01	0.9901	0.9896	0.989	0.985
0.1	0.906	0.902	0.898	0.867
0.2	0.824	0.818	0.811	0.761
0.5	0.632	0.624	0.615	0.547
1.0	0.432	0.425	0.417	0.357
2.0	0.245	0.241	0.236	0.198
5.0	0.100	0.098	0.096	0.080

TABLE 1. Phase velocity; $-u_{0\eta} = 1$, $\gamma = 2$, $\delta = 1$, $\Gamma = 1$

$k\backslash\alpha_0$	0.001	0.05	0.1	0.5
0.01	-1×10^{-9}	-4×10^{-9}	-8×10^{-8}	-1×10^{-7}
0.1	-7×10^{-7}	-3×10^{-5}	-6×10^{-5}	-8×10^{-5}
0.2	-4×10^{-6}	-2×10^{-4}	-3×10^{-4}	-4×10^{-4}
0.5	-2×10^{-5}	-1×10^{-3}	-2×10^{-3}	-2×10^{-3}
1.0	-5×10^{-5}	-2×10^{-3}	-4×10^{-3}	-5×10^{-3}
2.0	-6×10^{-5}	-3×10^{-3}	-5×10^{-3}	-7×10^{-3}
5.0	-6×10^{-5}	-3×10^{-3}	-5×10^{-3}	-7×10^{-3}

TABLE 2. Damping ω_{1I}

It is found that for the second mode, given by ω_2, the effect of finite α_0 is to introduce a small positive phase velocity and a small change in damping (and never an amplification).

In the physically realistic situation of small α_0, we can expand the roots of (3.16) in powers of α_0. The first two terms of the expansion of ω_{1R} and ω_{1I} (where $\omega_1 = \omega_{1R} + i\omega_{1I}$) are

$$\omega_{1R} = \frac{-u_{0\eta}}{1+\coth k\delta_0}\left\{1-\alpha_0\frac{\gamma-1}{\gamma}\left[\frac{1}{1+\coth k\delta_0}+\frac{(\gamma-1)\Gamma_0^2(1+\coth k\delta_0)}{\gamma^2u_{0\eta}^2+\Gamma_0^2(1+\coth k\delta_0)^2}\right]\right\}+O(\alpha_0^2), \tag{4.4}$$

$$\omega_{1I} = -\frac{\alpha_0(\gamma-1)^2u_{0\eta}^2\Gamma_0}{\{\gamma^2u_{0\eta}^2+\Gamma_0^2(1+\coth k\delta_0)^2\}\{1+\coth k\delta_0\}}+O(\alpha_0^2). \tag{4.5}$$

From (4.5) we note that for long waves, i.e. $|k\delta_0| \ll 1$, the damping is negligible, being proportional to $\alpha k^3\delta_0^3$.

This work was completed when the author was on a European Fellowship of the Alexander von Humboldt Foundation at the Technical University of Vienna. He sincerely thanks Professor W. Schneider and his colleagues for valuable discussions.

REFERENCES

DAVIS, R. H., HERBOLZHEIMER, E. & ACRIVOS, A. 1983 Wave formation and growth during sedimentation in narrow tilted channels. *Phys. Fluids* **26**, 2055–2064.

HERBOLZHEIMER, E. 1983 Stability of the flow during sedimentation in inclined channels. *Phys. Fluids* **26**, 2043–2054.

KYNCH, G. J. 1952 A theory of sedimentation. *Trans. Faraday Soc.* **48**, 166–176.

SARIC, W. S. & NAYFEH, A. H. 1975 *Phys. Fluids* **18**, 945–950.

SCHAFLINGER, U. 1983 Experimentelle und theoretische Untersuchungen zur Sedimentation in Behältern mit geneigten Wänden. Ph.D. thesis, T. U. Wien.

SCHNEIDER, W. 1982 Kinematic-wave theory of sedimentation beneath inclined walls. *J. Fluid Mech.* **120**, 323–346.

WALLIS, G. B. 1969 *One-dimensional Two-Phase Flows*. McGraw-Hill.